Menger's theorem, Max-flow-min-cut theorem, and the problem in the Mongolian TST 2011 test 3 problem 3

Donghui Kim

Thursday, June 21, 2018

Abstract

This article contains materials cut from the book [**?**] related to the problem 3, test 3 in Mongolian TST 2011.

A directed graph is said to be **strongly connected** if every vertex is reachable from every other points.

1. Prove that a digraph G is strongly connected iff there is at least one edge leaving each set $X \subseteq V(G)$, $X \neq \emptyset$, $X \neq V(G)$. [**?**, 47p.]

A directed graph G is said to be k-edge-connected between a and b if we need at least k edges to be removed to separate a and b. k-connected between a and b is defined in the similar manner.

Theorem 1 (Menger's theorem)**.** *Let G be a digraph and a, $b \in V(G)$. Then the followings holds.*

1. *There are k edge-disjoint (a, b) paths iff G is k-edge-connected between a and b*
2. *There are k independent (a, b) paths iff G is k-connected between a and b.*
3. *analogous statement hold for undirected graphs.*

Proof. 1. We will make use of induction on $|E(G)|$. If there is an $S \subseteq V(G)$ which defines a k-element (a, b)-cut and $|S| \geq 2$, $|V(G) - S| \geq 2$, then apply the induction hypothesis on the graphs obtained by contracting S and $V(G) - S$, respectively.

Suppose that G has k edge-disjoint (a, b)-paths, it is obviously k-edge-connected between a and b. To prove the other part, remove edges till the removal of any further edge will destroy k-edge-connectivity between a and b. Then obviously, there will be no edge with head at a or tail at

b. Assume first there is an edge e_i incident neither to a nor to b. Since $G-e_1$ no longer satisfies the conditions, it has a $(k-1)$-element (a,b)-cut C'. Then $C = C' \cup \{e_1\} = \{e_1, \ldots, e_k\}$ is a k-element (a,b)-cut by the choice of e_1, the set S determining C satisfies $|S| \geq 2$, $|V(G) - S| \geq 2$.

Let G_1, G_2 be the graphs obtained by contracting S and $V(G) - S$, respectively; let a' and b' be the images of a in G_1 and b in G_2, respectively. Obviously, G_1 is k-connected between a' and b, and thus by the induction hypothesis, there are k edge-disjoint (a',b)-path P_1, P_2, $\ldots$, P_k. Since the edges going out from a' are only e_1, $\ldots$, e_k, we may assume that $e_i \in P_i$. Similarly, there are k edge-disjoint (a,b')-paths Q_1, $\ldots$, Q_k in G_2, $e_i \in Q_i$. Then $P_1 \cup Q_1$, $\ldots$, $P_k \cup Q_k$ form k-edge-disjoint (a,b)-paths in G. What is left is the case, when each edge has tail at a or head at b. If there is an (a,b)-edge, we can remove it and proceed by induction on k, thus we may assume that there is no such edge. For any $x \neq a,b$, let $k(x)$ be the minimum of the numbers of (a,x)-edges and (x,b)-edges. Then obviously, there are $\sum_{x\neq a,b} k(x)$ edge-disjoint (a,b)-paths. On the other hand, let S be the set of all points x which are connected to b by $k(x)$ edges. Then, the cut determined by $\{a\} \cup S$ has exactly $\sum_{x\neq a,b} k(x)$ edges. Hence $\sum_{x\neq a,b} k(x) = k$ which proves the assertion.

2. Split each point $x \neq a$, b into two points x_1, x_2, where x_1 is joined to x_2 and x_2 is joined to y_1 iff x is joined to y.

   Consider a graph G', which has points a, b and two points x_1, x_2 that each $x \in V(G)$, $x \neq a$, b. Put $a_1 = a_2 = a$ and $b_1 = b_2 = b$. For any edge $e = (x,y) \in E(G)$, G' has the edge $e' = (x_2, y_1)$, moreover, for each $x \in V(G)$, $x \neq a, b$, it has the edge (x_1, x_2). Now

   (i) G' is k-edge-connected between a and b iff G is k-connected between a and b;

   (ii) G' has k-edge-disjoint (a,b)-paths iff G has k vertex-disjoint (a,b)-paths.

   To show (i), consider an (a,b)-cut C in G'. Let A consist of all points x such that $(x_1,x_2) \in C$ and all other edges of C. Then $|A| = |C|$ and A separates a and b in G; for if P is any (a,b)-path in G, then the edges of G' corresponding to edges and inner points of P, form an (a,b)-path P' in G and since P' contains an edge of C, P contains an edge or point of A.

   Conversely, if A is a set of edges and points separating a and b in G, then the construction above associates a set C of edges of G' with it, and C will be an (a,b)-cut with $|C| = |A|$. This proves (i).

   Now consider k edge-disjoint (a,b) paths P_1, $\ldots$, P_k in G'. If $x_i \in P_j$ then, obviously, (x_1,x_2) is an edge of P_j and x_i is also on P_j. Hence P_1, $\ldots$, P_k are vertex-disjoint (a,b)-paths of G.

Conversely, if there are k vertex-disjoint (a,b)-paths in G, then the (a,b)-paths of G' associated with them in the natural way are vertex-disjoint, and hence edge-disjoint. This proves (ii). Now (i) and (ii) proves the second statement by the first.

3. Replace each edge by two opposite oriented edges.

   Let directed graph $\vec{G}$ obtained as in the hint has the same connectivity and edge-connectivity between any two points as G. Moreover, the maximum number of edge-disjoint (vertex-disjoint) (a,b)-paths of G and $\vec{G}$ are the same. For edge(vertex) disjoint (a,b)-paths of $\vec{G}$, we may assume that they do not use both (x,y) and (y,x); for in this case one can easily find another system of the same number of (a,b)-paths, which contain neither (x,y) nor (y,x) (in the vertex-connectivity case, this difficulty does not even arise). These paths yield edge-disjoint (vertex-disjoint) (a,b)-paths in G.

□

Let $N=(V,E)$ be a directed graph. Let $s,t \in V$, $s \neq t$ and $c : E \longrightarrow \mathbb{R}^+$, $(u,v) \mapsto c(u,v)$. A (s,t) **flow** subject to $c(u,v)$ is a mapping $f : E \longrightarrow \mathbb{R}^+$, $(u,v) \mapsto f(u,v)$ with the following properties.

i. For all $(a,b) \in E$, $f(a,b) \leq c(u,v)$.

ii. For all $v \in V$, $v \neq s,t$,

$$\sum_{\{u:(u,v)\in E\}} f(u,v) = \sum_{\{u:(v,u)\in E\}} f(v,u)$$

Then the value of a flow is defined by

$$|f| = \sum_{v:(s,v)\in E} f(s,v)$$

Note that the value satisfies

$$|f| = \sum_{v:(v,t)\in E} f(v,t)$$

In this case c is called the capacity, s is called a source, and t is called a sink.

An (s,t) cut $C=(S,T)$ is a partition of V such that $s \in S$, $t \in T$. The cut set X_C is the edges that connect source part of the cut to the sink part;

$$X_C = (S \times T) \cap E$$

The capacity of an (s,t) cut is the total capacity of its edges;

$$c(S,T) = \sum_{(u,v)\in X_C} c(u,v)$$

Proposition 0.1. *Let $N = (V, E)$ be a directed graph with an (s, t) flow f. Let $C = (S, T)$ be an (s, t) cut. Then the value of f is given by*

$$|f| = \sum f(X_C) - \sum f(X_C^*)$$

where $C^* = (T, S)$

Proof. Let $C = (S, T)$. Consider

$$\sum_{x \in S} \left(\sum_{e=(x,y)} f(e) - \sum_{e=(y,x)} f(e) \right)$$

Since f is an (s, t) flow, the only non-zero term in the expression is

$$\sum_{e=(s,y)} f(e) - \sum_{e=(y,s)} f(e)$$

which is the value of f. On the other hand, each e spanned by S occurs twice with opposite signs, the edges of C occur with positive sign, the edges of C^* with negative sign. Hence

$$|f| = \sum f(X_C) - \sum f(X_C^*)$$

$\square$

Theorem 2 (Max-flow-min-cut theorem)**.** *Let $N = (V, E)$ be a directed graph with capacity c and $s, t \in V$. The maximum value of an (s, t) flow equals to the minimum capacity over all (s, t) cuts.*

Proof. The non-trivial part of the proof is to find an (s, t) flow f and an (s, t) cut C such that

$$|f| = \sum_{e \in C} c(e)$$

We will consider a flow f of maximum value subject to c. For each $e \in E$, we introduce a new edge e' having the same endpoints but converse orientation, and let

$$v_0(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e_1) & \text{if } e = {e_1}', e_1 \in E \end{cases}$$

We then consider the digraph G_1 determined by those edges e, e' for which $v_0(e) > 0$ and $v_0(e') > 0$; respectively. It will be shown that G_1 does not have (s, t)-path. To do this, suppose that there is (s, t)-path P in G_1. Let $\epsilon = min \sum_{e \in P} v_0(e) > 0$. Now set for $e \in E$,

$$f_1(e) = \begin{cases} f(e) + \epsilon & \text{if } e \in E(P) \\ f(e) - \epsilon & \text{if } e' \in E(P) \\ f(e) & \text{otherwise} \end{cases}$$

Then $f_1(e)$ is an (a,b)-flow of value $|f|+\epsilon$, a contradiction. Thus G_1 is not connected between s and t. Hence, by the problem 1, there is an $S \subseteq V$ such that $a \in S$, $b \notin S$ and the cut C of G determined by S satisfies

$$f(e) = c(e) \text{ if } e \in X_C,$$
$$f(e) = 0 \text{ if } e \in X_C^*$$

Hence, by proposition 0.1,

$$|f| = \sum_{e \in X_C} f(e) - \sum_{e \in X_C^*} f(e) = \sum_{e \in X_C} v(e)$$

Thus The value of max flow f is no less than the minimum value of the cut. On the other hand, for every (s,t) flow f subject to c and for any (s,t) cut C,

$$|f| = \sum_{e \in X_C} f(e) - \sum_{e \in X_C^*} f(e) \leq \sum_{e \in C} f(e) \leq \sum_{e \in C} c(e)$$

which completes the proof. □

2. Concerning a digraph, assume that the capacities are integers. Show that there is a maximum value flow with integral entries.

For a graph G and f be a non-negative integer valued function on $V(G)$ such that $f(v) \leq \deg(v)$ for every $v \in G$. An f factor of G is a subgraph of G with degree $f(v)$ for each $v \in V(G)$. When f is a constant function, say d, then the f factor is also called a d factor.

Theorem 3 (Ore-Gale-Ryser theorem)**.** *Let G be a bipartite graph with bipartition $\{A,B\}$ and $f(x) \geq 0$ an integer valued function on $V(G)$. Then G has an f factor if and only if*

i. $\sum f(A) = \sum f(B)$

ii. For all $X \subseteq A$ and $Y \subseteq B$,

$$\sum f(X) \leq m(X,Y) + f(B-Y)$$

where $m(X,Y)$ is the number of edges connecting X to Y.

Proof. Suppose that G has an f factor F. Then

$$\sum f(A) = |E(F)| = \sum f(B)$$

which proves i. Let $X \subseteq A$, $X \neq A$, $Y \subseteq B$. Then there are at most $m(X,Y)$ edges of F joining X to Y and at most $\sum_{y \in B-Y} f(y)$ edges of F joining X to $B-Y$. Since there are exactly $f(X)$ edges of F joining X to B, ii. follows.

Now suppose that i. and ii. are satisfied. Direct all edges from A to B. Take two points a, b and join a to each point of A and each point of B to b by directed edges. Define the capacity c over the resulring digraph G_0 by

$$c(e) = \begin{cases} f(x) & \text{if } e = (a,x) \text{ or } e = (x,b) \\ 1 & \text{if } e = (x,y),\ x \in A,\ y \in B \end{cases}$$

Observe that G_0 has an internal (a,b) flow with value $\sum f(A) = \sum f(B)$, if and only if G has an f factor. So what we have to verify is, by the max-flow-min-cut theorem (theorem 2) and problem 2, that each (a,b)-cut of G_0 has capacity at least $\sum_{x\in A} f(x)$.

Let S determine an (a,b)-cut $(a \in S \subset V(G_0))$. Set $X = S \cap A$ and $Y = B - S$. Then the capacity of the cut determined by S is

$$\sum_{x\in A-X} f(x) + \sum_{y\in B-Y} f(y) + m(X,Y) \geq \sum_{x\in A-X} f(x) + \sum_{x\in X} f(x) = \sum_{x\in A} f(x)$$

by (ii). This completes the proof. □

Now we are ready to solve the problem.

3. Let n and d be positive integers satisfying $d < \frac{n}{2}$. There are n boys and n girls in a school. Each boy has at most d girlfriends and each girl has at most d boyfriends. Prove that one can introduce some of them to make each boy have exactly $2d$ girlfriends and each girl have exactly $2d$ boyfriends.

Notes

1. Let G be a strongly connected digraph and X be a nonempty proper subset of $V(G)$ Let $a \in X$ and $b \in V(G)$. Then there is a path from a to b. Therefore, there is an edge leaving X.

Conversely, suppose that there is no path from a to b. Let X be a set of vertex reachable from a. Then by the maximality, there is no edge leaving X. □

2. Let us substitute $v(e)$ parallel edges for each edge e and let G_1 be the resulting graph. Let C be an (a,b)-cut of G and C_1 the (a,b)-cut of G_1, determined by the same set. Then

$$|C_1| = \sum_{e\in C} v(e)$$

Thus putting

$$V = \min_C \sum_{e\in C} v(e)$$

we get $|C_1| \geq V$ and hence, by Menger's theorem, we find V edge-disjoint (a,b)-path in G_1. Let $f(e)$ be the number of edges parallel to e used by these paths, then $f(e)$ is an (a,b)-flow of value V and with integral entries. This constructs integer valued flow. □

3. This is to prove the following.

> Let G be a simple bipartite graph with bipartition $\{A, B\}$ such that $|A| = |B| = n$, and with maximum degree $d < \frac{n}{2}$. Show that G can be embedded in a simple gegular bipartite graph on the same set $V(G)$ of points with degree $2d$.

Considering the bipartite complement $\widetilde{G}$ of the graph G, we should prove that $\widetilde{G}$ has an $n-2d$ factor.

By theorem 3, it suffices to show that for each $X \subseteq A$ and $Y \subseteq B$,

$$(n-2d)(n-|Y|) + m_{\widetilde{G}}(X, Y) \geq (n-2d)\,|X|$$

or equivalently,

$$(n-2d)\,(|X| + |Y| - n) \leq m_{\widetilde{G}}(X, Y)$$

Because this inequality is symmetric in X and Y, it suffices to consider the case $|X| \geq |Y|$. Noting that

$$m_{\widetilde{G}}(X, Y) = |X| \cdot |Y| - m_G(X, Y)$$

it suffices to show that

$$(n-2d)(|X| + |Y| - n) \leq |X| \cdot |Y| - m(X, Y)$$

where $m = m_G$. This would follow from

$$(|X| + d - n)(|Y| + 2d - n) \geq d(2d - n)$$

If $|Y| \leq n - 2d$, then $|Y| + 2d - n \leq 0$ and so,

$$(|X| + d - n)\,(|Y| + 2d - n) \geq (n + d - n)(|Y| + 2d - n) \geq d(2d - n)$$

If $n - 2d \leq |Y| \leq |X| \leq n - d$ and $|X| \geq d$, then $|X| + d - n \leq 0$, thus

$$(|X| + d - n)\,(|Y| + 2d - n) \geq (|X| + d - n)\,d \geq (2d - n)d$$

Finally, if $|X| \leq d$, we have

$$m_G(X, Y) \leq |X|\,|Y|$$

and thus it suffices to show that

$$(n-2d)\,(|X| + |Y| - n) \leq 0$$

which is clear, since $|Y| \leq |X| \leq d < \frac{n}{2}$ □

References

[Lov93] László Lovász. *Combinatorial Problems and Exercises.* AMS Chelsea Publishing, second edition, 1993.